Large Fluctuations in Stochastically Perturbed Nonlinear Systems: Applications in Computing

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1 Introduction

Nonlinear dynamical systems often display complex behavior. In this lecture I shall review the behavior of *stochastically perturbed* dynamical systems, which is a field of its own. I shall use this as an opportunity to discuss applications to computer science, though applications to statistical physics, chemical physics, and elsewhere in the sciences are also numerous.

If a deterministic dynamical system has an attractor, by definition the system state approaches the attractor in the long-time limit. But if the system is regularly subjected to small stochastic fluctuations (random kicks, or noise) this approach will only be approximate. In the long-time limit the system state will typically be specified by a probability distribution (a 'noisy attractor') centered on the attractor proper. In the limit as the noise strength tends to zero, this distribution will converge to the attractor.

Even if the system has a single globally stable point as its only attractor, one can pose an interesting question: What is the probability, if the noise strength is very small, of finding the system in a specified state macroscopically distant from the attractor? How long must one wait before this occurs? If the system has more than a single stable state, each with its own basin of attraction, one can similarly ask for the timescale on which transitions

*Supported in part by the National Science Foundation under grant NCR-90-16211.

between the two basins occur. Such questions are really questions about the character of the extreme tail of the noisy attractor, and can be answered only by quantifying the probability of *large fluctuations* of the system. The mathematical field dealing with such matters is known as large deviation theory [4, 26].

In scientific applications one would usually like to know not only how frequently atypical fluctuations occur, but also along which trajectory the system state moves during transitions from one stable state to another. It turns out that in most stochastically perturbed dynamical systems a single trajectory in the system state space, or at most a discrete set, is singled out in the limit of weak noise as by far the most likely.

This phenomenon has long been known to chemical and statistical physicists, but its importance in other fields which make use of stochastic modelling, such as ecology and evolutionary biology, has only recently become clear [8, 20]. In chemical physics the most likely transition trajectory is interpreted as a reaction pathway, since chemical reactions are modelled as transitions from a metastable state to a more stable state [25]. But the mathematical approach I shall sketch is much more general: the dynamical system can be continuous or discrete, and the system dynamics need not obey detailed balance. Some of the strongest results on systems without detailed balance have only recently been obtained [14, 15]. The system can even be *distributed*, with nontrivial spatial extent; this includes stochastic cellular automata, and systems specified by stochastic partial differential equations rather than stochastic ordinary differential equations.

The quasi-deterministic phenomena (optimal trajectories, well-defined reaction pathways, *etc.*) which arise in stochastically perturbed dynamical systems can be viewed as *emergent*. They are determined by the stochastic dynamics, but in a rather complicated way, and they manifest themselves only in the weak-noise limit. Their appearance in computer science applications is not well known; I hope the two examples treated in this lecture will correct that. Attempts have recently been made to interpret the behavior of computers, or interacting networks of computers, in dynamical system terms or even ecological terms [7]. But stochasticity is, I think, a crucial part of any such interpretation.

2 A Simple Stochastic Model: ALOHAnet

As a first example drawn from computer science, consider a stochastic model which attempts to capture the essential features of a large number of computers communicating with each other across a data network, such as an Ethernet. The model will be idealized, but it will be typical of ("in the same universality class as") models in which a large number of agents share occasional access to a single resource. Here the resource will be the network bus: the ether, which only one computer can use at a time.

You are no doubt familiar with such application programs as `telnet` and `ftp`, which allow a user of one machine to communicate with another. Behind the scenes ("at a lower protocol layer," in telecommunications jargon) these programs work as follows [24]. A connection between two computers consists of a stream of data packets, each typically containing between 10 and 10^3 bytes. (A data packet is simply a train of square waves.) An interactive login program like `telnet` normally transmits a packet whenever the user presses a key; the packet contains the typed character. Less interactive programs like `ftp`, which transfers whole files, employ larger packets. There is a scheme known as TCP/IP (Transmission Control Protocol/Internet Protocol) for specifying the destination of packets, and for keeping the two communicating computers synchronized. This last task may involve the transmission of additional packets.

Let us suppose that a computer is making substantial use of the network: several users are running `ftp` simultaneously, for example. In such a situation a statistical treatment is possible. In the context of a particular stochastic model, it is possible to estimate mean network usage, and the probability that data packets are transmitted successfully. That is what I shall now do.

A slight digression is necessary on the issue of *successful* transmission. Ethernet, besides being a tradename, is a multiaccess protocol: a scheme for sharing access to the cable connecting two or more computers. Normally when a computer wishes to transmit a packet, it does so immediately. It is possible therefore for two machines to transmit colliding packets, in which case both packets are corrupted: the information in both is lost. The Ethernet protocol (a CSMA/CD [Carrier Sense Multiple Access/Collision Detect] protocol) embodies a heuristic for minimizing the probability of collisions, *i.e.*, of unsucessful transmissions.

A description of the protocol may be found in the book of Bertsekas

and Gallager [1]. On grounds of simplicity I shall model a conceptually similar but simpler protocol known as ALOHAnet. ALOHAnet was one of several Ethernet precursors, developed at the University of Hawaii during the 1970's. Although it has long since been superseded, it lives on in the form of a tractable mathematical model. The stochastic ALOHAnet model is a discrete-time model or Markov chain, unlike the continuous-time models which must be employed in the performance analysis of real-world Ethernets. The following description is standard [5, 9, 13, 19].

Suppose that N computers are attached to the network; N will eventually be taken to infinity, yielding a continuum limit which (if proper scaling is imposed) can be viewed as a weak-noise limit. At each integer time $j = 1, 2, 3, \ldots$ a packet of data originates with probability p_0 on each computer not currently blocked. When is a computer blocked? When a previously generated packet has failed to be transmitted successfully, and the packet is awaiting retransmission.

Newly generated packets are always transmitted immediately, but of course they may collide with packets transmitted by other computers at the same integer time. Such collisions are immediately detected, and each of the transmitting computers enters a blocked state (if it was not blocked already). While in the blocked state, at each subsequent integer time a computer will attempt a retransmission with probability p_1. In other words each of the blocked computers backs off a random amount of time, and tries again to transmit its packet. The backoff time is geometrically distributed, with parameter p_1. This random backoff policy facilitates the breaking of the deadlock: if the blocked computers each backed off a *fixed* amount of time, they would simply run into each other again.

This ALOHAnet model has only three parameters: p_0, p_1, and N. If y_j is the number of computers blocked at time j, then $y_1, y_2, y_3 \ldots$ is a Markov chain on the discrete state space $\{0, 1, 2, \ldots, N\}$. Let us analyse this Markov chain.

At any time j, the number of retransmitted packets is binomially distributed, with parameters p_1 and y_j. Similarly, the number of newly generated (and transmitted) packets is binomially distributed with parameters p_0 and $N - y_j$. If X_1 and X_0 denote these two random variables, the total

number of packets transmitted at integer time j is $X_1 + X_0$, and

$$\xi \equiv y_{j+1} - y_j = \begin{cases} -1, & \text{if } X_0 = 0,\ X_1 = 1; \\ X_0, & \text{if } X_0 + X_1 > 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

y_j will decrease by 1 if a previously unsuccessfully transmitted packet (and only that packet) is retransmitted. It will increase by X_0 in the event of a collision, and so forth. From (1), it is easy to work out the density of the random variable $\xi \equiv \Delta y$.

Since we wish to construct a continuum large-N limit we define the normalized network state x at any time to be y/N, the fraction of computers that are currently blocked. Necessarily $0 \leq x \leq 1$. Besides scaling the state space in this way, we scale time by defining normalized time t to equal j/N, so that x, if viewed as a function of t, jumps at $t = 1/N, 2/N, \ldots$ by a random quantity $N^{-1}\xi$. The density of the random variable ξ is specified by the current normalized state x; we write ξ as $\xi(x)$ to make this clear.

To get a nontrivial large-N limit we need to scale the probabilities p_0 and p_1 as well; we take $p_0 = q_0/N$ and $p_1 = q_1/N$, for some N-independent q_0 and q_1. So $q_0 x$ is the expected number of newly generated packets, and $q_1(1-x)$ the expected number of retransmitted packets, at any specified normalized time j/N. It is an easy exercise to verify that in the large-N limit

$$\langle \xi(x) \rangle = q_0(1-x) - [q_0(1-x) + q_1 x] \exp\left[-q_0(1-x) - q_1 x\right] \tag{2}$$

is the expected change in the number of blocked computers, at any specified time j/N. The formula (2) gives us an explicit expression for $\langle \Delta x \rangle$, the mean amount by which the normalized state x changes at any specified time j/N; it is simply $N^{-1}\langle \xi(x) \rangle$. So in the large-$N$ limit the dynamics of our network model are *in expectation* completely specified by (2).

We can now see how the ALOHAnet model can be viewed as a stochastically perturbed dynamical system. In expectation, the large-N ALOHAnet model looks very like a one-dimensional dynamical system

$$\dot{x}(t) = \langle \xi(x) \rangle. \tag{3}$$

defined on the closed interval $[0, 1]$. Such an associated deterministic dynamical system is called a *fluid approximation* by network performance analysts.

Although (as we shall see) it cannot answer the questions about large fluctuations in which we are interested, the fluid approximation says quite a bit about the stability of the network. In Fig. 1, the drift field $\langle \xi(x) \rangle$ is plotted as a function of x, for $q_0 = 0.43$ and $q_1 = 5.0$ (parameter values originally chosen by Günther and Shaw [5]). It is clear that for this choice of parameters the system has two point attractors: $x_0 \approx 0.150$ and $x_1 \approx 0.879$. Each has its own basin of attraction, and in the fluid approximation the network state flows deterministically to one or the other. The two attractors are interpreted as follows. Networks, in particular heavily loaded networks, are prone to *congestion*, and the two attractors are respectively a low-congestion and a high-congestion state.

The presence of more than a single attractor, for certain parameter values, is an unfortunate feature of the ALOHAnet protocol. If at time zero all computers begin unblocked, with these parameter values the fraction of blocked computers will swiftly rise to ≈ 0.150. If on the other hand at time zero the computers all begin in the blocked state, the fraction will decrease to ≈ 0.879 and no further. In the latter case very few packets are successfully transmitted or retransmitted, since the probability of more than a single computer transmitting a packet is always very high. (Since $q_1 = 5.0$, when $x \approx 1$ about 5 computers, on average, attempt to retransmit a packet at each time j/N.) The ALOHAnet protocol makes no provision for breaking the deadlock by sharing the network in a sequential or round-robin fashion: in the event of extreme congestion, the computers get in each others' way.

The appearance of more than a single point attractor is actually a bit atypical; it will occur only for certain values of the scaled parameters. (See Fig. 2.) The (q_0, q_1)-plane is divided into two regions: a monostable (one-attractor) region, and a bistable (two-attractor) region. The equilibrium blocking fraction is a single-valued function of (q_0, q_1) in the former region, and a double-valued function in the latter. Nelson [18] has shown that this phenomenon, which is so suggestive of statistical-mechanical critical behavior, generalizes naturally to multidimensional parameter spaces. The Ethernet protocol modifies the packet retransmission probability each time an unsuccessful retransmission occurs, so a more realistic ALOHAnet model would be specified by a vector $(p_0, p_1, p_2, \ldots)$ of probabilities, with p_k, $k \geq 1$, the probability of transmitting a packet which has failed to be successfully transmitted exactly k times. The corresponding normalized system state would be a vector $(x^{(1)}, x^{(2)}, \ldots)$ of blocking fractions: $x^{(k)}$, $k \geq 1$, would be the fraction of com-

puters which are blocked and which have failed to transmit a stored packet exactly k times. The analogue of Fig. 2 would be a multidimensional phase diagram, some regions in which would be characterized by the presence of multiple point attractors in the multidimensional normalized state space.

The preceding treatment has been entirely in the context of the deterministic fluid approximation. The network state does not actually evolve deterministically, except in expectation. The expected increment $\langle \Delta x \rangle$ equals $N^{-1}\langle \xi(x) \rangle$, but the standard deviation of Δx is also proportional to N^{-1}. Δx equals $\langle \Delta x \rangle$ plus $\Delta x - \langle \Delta x \rangle$, and the latter term can be viewed as a stochastic perturbation superimposed on the dynamical system. These stochastic perturbations will broaden the point attractors into noisy attractors, and occasionally induce transitions between them.

These transitions are of considerable practical interest, since they are sudden changes in network congestion. A heavily loaded network can suddenly shift from a low-congestion state to a high-congestion state, in which almost no packets are transmitted successfully. (This has rather drastic effects on the computers attached to the network!) But to model such transitions, a fully stochastic treatment is necessary.

3 Wentzell-Freidlin Theory

The techniques employed to estimate the transition time between metastable states, and in general to estimate the probability of unlikely events in the weak-noise limit, go under the name of Wentzell-Freidlin theory [26]. Wentzell-Freidlin theory is simply the large deviation theory of stochastically perturbed dynamical systems. Many results in this area are due to physicists and chemists [6, 23, 25], but Wentzell and Freidlin were the first to put the subject on a sound mathematical footing [4, 27]. I shall summarize their main results, and extensions.

Consider a multidimensional random process $\mathbf{x}(t)$ similar to the normalized ALOHAnet process. $\mathbf{x}(t)$ is assumed to jump at times $t = N^{-1}, 2N^{-1}, 3N^{-1}, \ldots$, and the jump magnitude is N^{-1} times a random vector whose distribution depends on the current state $\mathbf{x}$. We write this random vector as $\xi(\mathbf{x})$, so $\Delta\mathbf{x} = N^{-1}\xi(\mathbf{x})$. The $N \to \infty$ limit will be a weak-noise limit.

This random process strongly resembles a diffusion process with drift. In fact the expected drift velocity at any point $\mathbf{x}$ is $\mathbf{u}(\mathbf{x}) \equiv \langle \xi(\mathbf{x}) \rangle$, and

the diffusion tensor is N^{-1} times $D_{ij}(\mathbf{x}) \equiv \mathrm{Cov}(\xi_i(\mathbf{x}), \xi_j(\mathbf{x}))$, the covariance matrix of the components of $\xi(\mathbf{x})$. A continuous-time diffusion process $\mathbf{x}(t)$ with these parameters would satisfy the stochastic differential equation

$$dx_i(t) = u_i(\mathbf{x}(t)) + \sum_j \frac{\sigma_{ij}(\mathbf{x}(t))}{\sqrt{N}} dw_j(t) \tag{4}$$

where $d\mathbf{w}(t)$ is white noise, and the tensor $\sigma = (\sigma_{ij})$ is related to the tensor $\mathbf{D} = (D_{ij})$ by $\mathbf{D} = \sigma\sigma^t$. But this continuous-time 'diffusive approximation' to the underlying jump process is not especially useful for our purposes: the large fluctuations of the jump process turn out to depend crucially on the higher moments of $\xi(\mathbf{x})$.

Suppose that $\mathbf{x}_0$ is an attractor for the expected drift field $\mathbf{u}(\mathbf{x})$. Then in expectation $\mathbf{x}(t)$ will tend to flow toward $\mathbf{x}_0$ if it begins in the basin of attraction of $\mathbf{x}_0$. Thereafter, $\mathbf{x}(t)$ will tend to wander near $\mathbf{x}_0$ for a long time. But statistical fluctuations of all magnitudes will occur; the stochastic perturbations $N^{-1}[\xi(\mathbf{x}) - \mathbf{u}(\mathbf{x})]$ will eventually push $\mathbf{x}$ outside any specified region U surrounding $\mathbf{x}_0$. In other words, the noise will eventually overcome the drift.

Since the effective diffusion coefficient decays as N^{-1}, one expects that the time to exit any specified region U grows (in expectation) exponentially in N. That is correct, and Wentzell-Freidlin theory provides a technique for computing the asymptotic exponential growth rate. This will of course depend on the choice of U. In most applications U is the entire basin of attraction of the attractor $\mathbf{x}_0$, though a smaller region could be chosen.

The technique is as follows. According to theory the expected exit time $\langle t_{\mathrm{exit}} \rangle$ has weak-noise asymptotics

$$\langle t_{\mathrm{exit}} \rangle \sim \exp(N\mathcal{S}_0), \qquad N \to \infty \tag{5}$$

where

$$\mathcal{S}_0 = \inf \int L(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \tag{6}$$

is a *minimum action* for exiting trajectories. The infimum is taken over all trajectories $\mathbf{x}(t)$ which begin at $\mathbf{x}_0$ and terminate on the boundary of U. The transit time is left unspecified. Here $L(\mathbf{x}, \dot{\mathbf{x}})$ is a Lagrangian function, dual to a Hamiltonian or energy function constructed from the distribution of $\xi(\mathbf{x})$ by the formula

$$H(\mathbf{x}, \mathbf{p}) = \log\langle \exp(\mathbf{p} \cdot \xi(\mathbf{x})) \rangle. \tag{7}$$

It is clear that the higher moments of $\xi(\mathbf{x})$ enter into the computation of the function H. $H(\mathbf{x}, \cdot)$ is in fact the cumulant generating function of the random variable $\xi(\mathbf{x})$.

The sudden appearance of a classical Hamiltonian and its dual Lagrangian is quite remarkable. They are not mere mathematical auxiliaries. The trajectory $\mathbf{x}^*(t)$ minimizing the action (it usually exists, and is unique) is interpreted as the *most probable exit path* (MPEP) in the limit of weak noise. It is not difficult to check, using standard methods of classical mechanics, that the optimization of the action over transit times yields an MPEP which is a *classical trajectory of zero energy*. So the 'momentum' $\mathbf{p}$, which has no direct physical interpretation, as a function of position $\mathbf{x}$ along the MPEP must satisfy

$$\langle \exp(\mathbf{p} \cdot \xi(\mathbf{x})) \rangle = 1. \tag{8}$$

If the state space is one-dimensional, this zero-energy constraint alone will determine the MPEP.

The MPEP $\mathbf{x}^*$ is not only a most probable exit path: it is also an exit path of least resistance. Although $\mathbf{x}(t)$ will remain in U for an exponentially long time, it will fluctuate out along the MPEP (and in other directions) an exponentially large number of times before the MPEP is traversed in full and U is exited. The final fluctuation will follow $\mathbf{x}^*$ quite closely in the large-N limit. One can view the equilibrium distribution of the system state $\mathbf{x}$ (the noisy attractor) as being concentrated near $\mathbf{x}_0$, but having a tube-like protuberance stretching out toward the boundary of U along the trajectory $\mathbf{x}^*$. In the large-N limit the tube is exponentially suppressed, and the noisy attractor converges to the point attractor $\mathbf{x}_0$.

$\langle t_{\text{exit}} \rangle$ grows exponentially in N, but the limiting *distribution* of t_{exit} has not yet been specified. It turns out to be an exponential distribution. This is very typical of weak-noise escape problems, where the probability of any single escape attempt is small. (The same exponential distribution is seen in radioactive decay.)

$\mathcal{S}_0$, the weak-noise growth rate of the expected exit time, can be viewed as a *barrier height*: a measure of how hard it is to overcome the drift driving $\mathbf{x}$ toward $\mathbf{x}_0$ and away from the boundary of U. In fact the Wentzell-Freidlin framework, if extended to conservative continuous-time processes described by (4), yields the familiar Arrhenius law for the growth of the exit time in the limit of weak noise. For such systems $\mathcal{S}_0$ is simply the height of the potential

barrier surrounding the attractor.

What is not clear from the Wentzell-Freidlin treatment (and is still not *rigorously* clear, though numerous nonrigorous results have been obtained [14, 16, 17]) is the subdominant large-N asymptotics of $\langle t_{\text{exit}} \rangle$. In general one expects

$$\langle t_{\text{exit}} \rangle \sim C N^{\alpha} \exp(N \mathcal{S}_0), \qquad N \to \infty, \tag{9}$$

for some constants C and α, but Wentzell-Freidlin theory yields only the exponential growth rate $\mathcal{S}_0$. The pre-exponential factor in (9) remains to be determined.

The current status of the prefactor problem can be summed up as follows. If U is taken to be the entire basin of attraction of $\mathbf{x}_0$, α is typically zero, and C can be obtained by a method of matched asymptotic expansions, *i.e.*, a method of systematically approximating the equilibrium distribution of $\mathbf{x}$. However in multidimensional models there is an entire zoo of possible pathologies, including the appearance of caustics and other singular curves in the state space [2, 14, 15], which can induce a nonzero α and/or hinder a straightforward computation of C. This is the case, at least, for continuous-time diffusion processes defined by stochastic differential equations. The situation for jump processes is expected to be similar.

4 Applying the Theory

Wentzell-Freidlin theory, with extensions, can be applied to the stochastic ALOHAnet model, and to other stochastically perturbed dynamical systems arising in computer science. The quantity most readily computed is $\mathcal{S}_0$, the exponential growth rate in the weak-noise limit of the expected time before the system leaves a specified region surrounding a point attractor in the system state space. Recall that in the ALOHAnet model this region is the basin of attraction; departure from it signals a drastic change in network congestion.

If the system state space is one-dimensional, as in the ALOHAnet model, the classical-mechanical interpretation of $\mathcal{S}_0$ facilitates its computation. $\mathcal{S}_0$ is always the action of a zero-energy trajectory, with energy as a function of position and momentum given by the formula (7). This Hamiltonian is a convex function of $\mathbf{p}$ at fixed $\mathbf{x}$, so if the state space is one-dimensional (and $\langle \xi(\mathbf{x}) \rangle \neq \mathbf{0}$, which will always be the case within the basin of attraction)

the equation $H(\mathbf{x}, \mathbf{p}) = 0$ will have only two solutions for $\mathbf{p} = \mathbf{p}(\mathbf{x})$. One of these is $\mathbf{p} \equiv \mathbf{0}$, which is unphysical. This solution is unphysical because if $\mathbf{p} = \mathbf{0}$

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\langle \xi(\mathbf{x}) \exp(\mathbf{p} \cdot \xi(\mathbf{x})) \rangle}{\langle \exp(\mathbf{p} \cdot \xi(\mathbf{x})) \rangle} = \langle \xi(\mathbf{x}) \rangle \tag{10}$$

and the $\mathbf{p} \equiv \mathbf{0}$ trajectory simply follows the mean drift, which points *toward* the attractor rather than away. The MPEP must be a classical trajectory emanating from the attractor, so in a one-dimensional system it is uniquely characterized by the condition that $\mathbf{p} = \mathbf{p}(\mathbf{x})$ be the nonzero solution of $H(\mathbf{x}, \mathbf{p}) = 0$. Actually there are two such trajectories, one emanating to either side of the attractor; the true MPEP will be the one with lesser action.

In general to compute $\mathcal{S}_0$, even in higher-dimensional models one needs only the MPEP and the momentum as a function of position along it. This is because the action of any zero-energy classical trajectory may be written as a line integral of the momentum, so that

$$\mathcal{S}_0 = \int \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} \tag{11}$$

the integral being taken along the MPEP from the attractor to the boundary of the region. But only in one-dimensional models is (11) easy to apply. In d-dimensional models merely finding the MPEP requires an optimization over the $(d-1)$-dimensional family of zero-energy trajectories extending to the boundary. Except in models with symmetry, this optimization must usually be performed numerically.

4.1 The ALOHAnet Application

In the ALOHAnet model, the expected drift $\langle \xi(x) \rangle$ as a function of normalized network state x is given by (2). But to study large fluctuations, and compute the MPEP, one needs the Wentzell-Freidlin Hamiltonian $\log\langle \exp(p\xi(x)) \rangle$. In the large-$N$ limit the random variables X_1 and X_0, in terms of which ξ is expressed by (1), become respectively a Poisson random variable with parameter $q_1 x$ and a Poisson random variable with parameter $q_0(1-x)$. A bit of computation yields

$$H(x,p) = \log\left[e^{q_0(1-x)(e^p-1)} + q_0(1-x)e^{-q_0-q_1x}(1-e^p) + q_1 x e^{-q_0-q_1x}(e^{-p}-1) \right] \tag{12}$$

as the Hamiltonian.

If the parameters q_0 and q_1 are known, it is easy to compute the momentum $p = p(x)$ along the MPEP, by numerically solving for the nonzero solution of the implicit equation $H(x, p(x)) = 0$. But the MPEP, and hence $\mathcal{S}_0$, will depend on the choice of basin of attractor. With the parameter values $q_0 = 0.43$ and $q_1 = 5.0$ of Fig. 1, the two attractors $x_0 \approx 0.150$ and $x_1 \approx 0.879$ have respective basins of attraction $[0, x_c)$ and $(x_c, 1]$, with $x_c \approx 0.278$ the intermediate repellor. MPEPs extend from x_0 to x_c, and from x_1 to x_c. Numerical integration of $p(x)$ gives

$$\mathcal{S}_0[x_0 \to x_c] \approx 0.00177 \tag{13}$$

$$\mathcal{S}_0[x_1 \to x_c] \approx 0.014 \tag{14}$$

as the growth rates of the expected transition times.

We see that for the stochastically modelled ALOHAnet, in the large-N limit a reduced description is appropriate. Asymptotically, it becomes a *two-state process*. The network is either in a low-congestion state (the basin of attraction of x_0) or a high-congestion state (the basin of attraction of x_1), and the transition rates between them (the reciprocals of the expected transition times) display exponential falloffs

$$\exp\left(-N\mathcal{S}_0[x_0 \to x_c]\right), \qquad \exp\left(-N\mathcal{S}_0[x_1 \to x_c]\right) \tag{15}$$

respectively. With the above choice of parameters, for reasonable-sized N the latter transition rate is much smaller than the former. The network, once congestion has interfered with the proper performance of the backoff algorithm, gets 'stuck' for potentially a long time. This is clearly not a good choice of network parameters!

In a real-world N-computer ALOHAnet implementation, q_0 would be the total network load, and would be determined by the level of interprocessor computing taking place on the network. The backoff parameter $q_1 = Np_1$ however would probably be fixed, with p_1 hardcoded in a data communications chip installed in each computer. So the Wentzell-Freidlin approach could be employed to determine the likelihood, as a function of network load, of irreversible (or all but irreversible) congestion occurring.

Of course the bistability of the system is itself a function of q_0 and q_1. As noted, for many values of the parameters the network is monostable: there is only a single attractor, which may be characterized by a comparatively low

level of congestion. For such a network one could compute an action $\mathcal{S}_0$ for any specified maximum tolerable congestion level. The associated optimal (*i.e.*, most probable) approach path would be computed much as the MPEP is computed in the bistable case.

4.2 A Colliding Stacks Application

There have been several applications of large deviation theory to the stochastic modelling of *dynamic data structures* [10, 11, 12]. The memory usage of a program or programs being executed by a computer can be modelled as a discrete-time jump process. In many cases this process may be viewed as a finite-dimensional dynamical system, subject to small stochastic perturbations. Of interest is the amount of time expected to elapse before a particularly large fluctuation away from a deterministic point attractor occurs. This would correspond, in real-world terms, to an atypical string of memory allocations leading to an exhaustion of memory.

The following two-dimensional 'colliding stacks' model was first studied by Flajolet [3], having been first suggested by Knuth. Suppose that N cells of memory, arranged in a linear array, are available for use by two programs. Suppose that at any given time, the programs will require $y^{(1)}$ and $y^{(2)}$ cells of memory respectively. It will be most efficient for them to employ respectively the first $y^{(1)}$ and the last $y^{(2)}$ cells of the array, so as to avoid contention for memory. It is necessary that $y^{(1)} + y^{(2)} \leq N$; if this inequality becomes an equality, the two-program system runs out of memory.

A natural model for the evolution of $y^{(1)}$ and $y^{(2)}$ is as follows. At any integer time $j = 1, 2, 3, \ldots$, there are four possibilities: $y^{(1)}$ may increase by 1, $y^{(1)}$ may decrease by 1, $y^{(2)}$ may increase by 1, and $y^{(2)}$ may decrease by 1. These are assigned probabilities $p/2$, $(1-p)/2$, $p/2$, $(1-p)/2$, for p the probability of a net increase in memory usage. Let us take $0 < p < \frac{1}{2}$, so that deallocations of memory are more likely than new allocations. (Note that if $y^{(1)} = 0$ or $y^{(2)} = 0$ the assigned probabilities must differ, since neither $y^{(1)}$ nor $y^{(2)}$ can go negative.)

Just as in the ALOHAnet model, it is natural to scale both time and and the state space as the amount of memory N tends to infinity. However, we shall not need to scale the model parameter p. Let $\mathbf{x} = (x_1, x_2) = (y^{(1)}, y^{(2)})/N$ be the normalized state of the two-program system, and let $t = j/N$ be normalized time. $\mathbf{x}$ jumps at $t = 1/N, 2/N, 3/N, \ldots$ by an

amount $N^{-1}\xi$, where ξ is a random variable with discrete density

$$\mathbf{P}\left\{\xi = \mathbf{z}\right\} = \begin{cases} p/2, & \text{if } \mathbf{z} = (1,0); \\ p/2, & \text{if } \mathbf{z} = (0,1); \\ (1-p)/2, & \text{if } \mathbf{z} = (-1,0); \\ (1-p)/2, & \text{if } \mathbf{z} = (0,-1). \end{cases} \tag{16}$$

As defined, the density of ξ is essentially independent of $\mathbf{x}$. It is useful to relax this assumption, so as to permit more realistic stochastic modelling of dynamic data structures.

$$\mathbf{P}\left\{\xi(\mathbf{x}) = \mathbf{z}\right\} = \begin{cases} p(x_1)/2, & \text{if } \mathbf{z} = (1,0); \\ p(x_2)/2, & \text{if } \mathbf{z} = (0,1); \\ (1-p(x_1))/2, & \text{if } \mathbf{z} = (-1,0); \\ (1-p(x_2))/2, & \text{if } \mathbf{z} = (0,-1) \end{cases} \tag{17}$$

is a natural generalization. Here $p(x)$ (assumed to take values between 0 and $\frac{1}{2}$ exclusive) specifies the probability of an increase in memory usage by either program, as a function of the fraction of available memory which that program is currently using. We now write ξ as $\xi(\mathbf{x})$, to indicate the dependence of its density on $\mathbf{x}$.

The normalized state $\mathbf{x}$ is confined to the right triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The expected drift

$$\langle \xi(\mathbf{x}) \rangle = (p(x_1) - \tfrac{1}{2}, p(x_2) - \tfrac{1}{2}) \tag{18}$$

may be viewed as a deterministic dynamical system on this two-dimensional normalized state space. Clearly, the vertex $(0,0)$ is the global attractor. In this model the two programs tend on the average not to use much memory.

Since there is only a single attractor, the quantity of interest is the expected time which must elapse before a fluctuation of specified magnitude occurs. Fluctuations which take the system state to the hypotenuse of the triangle (where $x_1 + x_2 = 1$, or $y^{(1)} + y^{(2)} = N$) are *fatal*: they correspond to memory exhaustion. The rate at which they occur can be estimated in the large-N limit.

This is a two-dimensional system, so the optimal (least-action) trajectories are not determined uniquely by the zero-energy constraint. However we still have

$$\langle t_{\text{exit}} \rangle \sim \exp(N\mathcal{S}_0), \qquad N \to \infty \tag{19}$$

with $\mathcal{S}_0$ the action of the least-action trajectory which exits the triangle through the hypotenuse. The action is computed from the Lagrangian dual to the Wentzell-Freidlin Hamiltonian

$$\begin{aligned} H(\mathbf{x},\mathbf{p}) &= \log\langle\exp(\mathbf{p}\cdot\xi(\mathbf{x}))\rangle \\ &= -\log 2 + \log\{\cosh p_x - [1-2p(x)]\sinh p_x \\ &\qquad + \cosh p_y - [1-2p(y)]\sinh p_y\}, \end{aligned} \tag{20}$$

which follows from (17).

The zero-energy trajectories determined by (20) are studied at length in Ref. [11], where it is shown that the MPEP depends strongly on the behavior of the function $p(x)$. (See Fig. 3.) If $p(x)$ is a strictly decreasing function, so that the model is 'increasingly contractive,' with large excursions away from the attractor strongly suppressed, then the MPEP turns out to be directed along the line segment from $(0,0)$ to $(\frac{1}{2},\frac{1}{2})$. Its action is

$$\mathcal{S}_0 = 4\int_{x=0}^{1/2} \tanh^{-1}[1-2p(x)]\,dx. \tag{21}$$

If on the other hand $p(x)$ is a strictly increasing function, so that the model is decreasingly contractive, with large excursions less strongly suppressed, then there is a twofold degeneracy. MPEPs are directed outward from $(0,0)$ to the two other vertices of the triangle, and

$$\mathcal{S}_0 = 2\int_{x=0}^{1} \tanh^{-1}[1-2p(x)]\,dx \tag{22}$$

is their common action.

So when $p(x)$ is strictly increasing, there is a 'hot spot' on the hypotenuse of the triangle at $(\frac{1}{2},\frac{1}{2})$. When the two-program system runs out of memory, as $N\to\infty$ it is increasingly likely that each program will be using approximately $N/2$ memory cells. If on the other hand $p(x)$ is strictly decreasing, there are hotspots at the vertices $(0,1)$ and $(1,0)$. Exhaustion increasingly tends to occur when one or the other program is using all, or nearly all, of the N memory cells.

If $p(x)$ is neither strictly increasing nor strictly decreasing, the large-N asymptotics may become more complicated. The most easily treated case is that of $p(x)\equiv p$, a constant, *i.e.*, the model of (16). In this model an *infinite degeneracy* occurs: any trajectory which moves some distance (possibly

zero) from $(0,0)$ toward $(0,1)$ or $(1,0)$, and then moves into the interior of the triangle at a 45° angle until it reaches the hypotenuse, is a least-action trajectory. Large fluctuations away from the attractor may proceed along any of this uncountable set of MPEPs. As a consequence there is no hotspot: in the large-N limit, the exit location is uniformly distributed over the hypotenuse. Flajolet [3] first discovered this phenomenon combinatorially, but it has a natural classical-mechanical interpretation. It is however a bit counterintuitive: it says that when memory is exhausted, the fractions allocated to each program are as likely to be small as large. This is a very sensitive phenomenon.

5 Conclusions

We have seen that the Wentzell-Freidlin results on scaled jump processes throw considerable light on the fluctuations of stochastically perturbed dynamical systems, in the weak-noise limit. The appearance of a classical Hamiltonian and Lagrangian, even if the unperturbed dynamical system is in no sense Hamiltonian, is quite striking. So is the central importance of zero-energy trajectories.

In this lecture I have focused on jump processes since they are the most relevant to computer science applications. (Computing is inherently discrete.) But they also occur in chemical physics: there is always an integer number of molecules in any given region of space. Attempts are now being made to interpret the stochastic aspects of chemical reactions in terms of optimal trajectories [21]. This is very reminiscent of our focus on most probable exit paths (MPEPs).

There is also a large deviation theory of continuous-time processes [4, 26], such as the diffusion processes specified by the stochastic differential equation (4). Associated to each such process is a Fokker-Planck equation (a parabolic partial differential equation) describing the diffusion of probability. The zero-energy classical trajectories of continuous-time large deviation theory can be viewed as the *characteristics* of this differential equation. Normally one expects only hyperbolic equations to have characteristics, but these characteristics are emergent: they manifest themselves only in the weak-noise limit.

A large deviation theory of spatially extended systems would be an in-

teresting extension, but is still under development. Such systems include stochastic partial differential equations and stochastic cellular automata. In such systems a MPEP would be a trajectory in the system state space, describing a most probable *spatially extended* fluctuation leading from one metastable state to another. Much work has been done on this by statistical mechanicians and field theorists (who call such fluctuations 'instantons' [22]), but the theory is less complete than the theory I have sketched in this lecture. The theory of extended fluctuations has in particular not been applied to distributed computer systems. There is clearly much left to be done!

References

[1] D. Bertsekas and R. Gallager. *Data Networks.* Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

[2] V. A. Chinarov, M. I. Dykman, and V. N. Smelyanskiy. Dissipative Corrections to Escape Probabilities of Thermally Nonequilibrium Systems. Preprint.

[3] P. Flajolet. The Evolution of Two Stacks in Bounded Space and Random Walks in a Triangle. In *Mathematical Foundations of Computer Science: Proceedings of the 12th Symposium* (Bratislava, Czechoslovakia, 1986), pp. 325–340. Springer-Verlag, New York/Berlin, 1986.

[4] M. I. Freidlin and A. D. Wentzell. *Random Perturbations of Dynamical Systems.* Springer-Verlag, New York/Berlin, 1984.

[5] N. J. Günther and J. G. Shaw. Path Integral Evaluation of ALOHA Network Transients. *Inf. Process. Lett.* **33** (1990), 289–295.

[6] P. Hänggi, P. Talkner, and M. Borkovec. Reaction-rate Theory: Fifty Years After Kramers. *Rev. Modern Phys.* **62** (1990), 251–341.

[7] B. A. Huberman, editor. *The Ecology of Computation.* Elsevier, New York/Amsterdam, 1988.

[8] R. Lande. Expected Time for Random Genetic Drift of a Population Between Stable Phenotypic States. *Proc. Nat. Acad. Sci. USA* **82** (1985), 7641–7645.

[9] J.-T. Lim and S. M. Meerkov. Theory of Markovian Access to Collision Channels. *IEEE Trans. Comm.* **COM-35** (1987), 1278–1288.

[10] G. Louchard and R. Schott. Probabilistic Analysis of Some Distributed Algorithms. *Random Structures and Algorithms* **2** (1991), 151–186. A preliminary version appeared in *Proceedings of CAAP '90*, Lecture Notes in Computer Science #431, pp. 177–190. Springer-Verlag.

[11] R. S. Maier. Colliding Stacks: A Large Deviations Analysis. *Random Structures and Algorithms* **2** (1991), 379–420.

[12] R. S. Maier. A Path Integral Approach to Data Structure Evolution. *J. Complexity* **7** (1991), 232–260.

[13] R. S. Maier. Communications Networks as Stochastically Perturbed Nonlinear Systems: A Cautionary Note. In *Proceedings of the 30th Allerton Conference on Communication, Control and Computing*, pp. 674–681, Monticello, Illinois, 1992.

[14] R. S. Maier and D. L. Stein. Transition-rate Theory for Non-Gradient Drift Fields. *Phys. Rev. Lett.* **69** (1992), 3691–3695.

[15] R. S. Maier and D. L. Stein. The Escape Problem for Irreversible Systems. Submitted to *Physical Review E*, 1993.

[16] B. J. Matkowsky, Z. Schuss, C. Knessl, C. Tier, and M. Mangel. Asymptotic Solution of the Kramers-Moyal Equation and First-Passage Times for Markov Jump Processes. *Phys. Rev. A* **29** (1984), 3359–3369.

[17] T. Naeh, M. M. Kłosek, B. J. Matkowsky, and Z. Schuss. A Direct Approach to the Exit Problem. *SIAM J. Appl. Math.* **50** (1990), 595–627.

[18] R. Nelson. The Stochastic Cusp, Swallowtail, and Hyperbolic Umbilic Catastrophes as Manifest in a Simple Communications Model. In *Performance '84*, edited by E. Gelenbe, pp. 207–224. North-Holland, 1984.

[19] R. Nelson. Stochastic Catastrophe Theory in Computer Performance Modeling. *J. Assoc. Comput. Mach.* **34** (1987), 661–685.

[20] C. M. Newman, J. E. Cohen, and C. Kipnis. Neo-Darwinian Evolution Implies Punctuated Equilibria. *Nature* **315** (1985), 400–401.

[21] J. Ross, K. L. C. Hunt, and P. M. Hunt. Thermodynamic and Stochastic Theory for Nonequilibrium Systems with Multiple Reactive Intermediaries. *J. Chem. Phys.* **96** (1992), 618–629.

[22] L. S. Schulman. *Techniques and Applications of Path Integration.* Wiley, New York, 1981.

[23] Z. Schuss. *Theory and Application of Stochastic Differential Equations.* Wiley, New York, 1980.

[24] W. Stallings. *Data and Computer Communications.* Macmillan, New York, second edition, 1988.

[25] N. G. van Kampen. *Stochastic Processes in Physics and Chemistry.* North-Holland, New York/Amsterdam, 1981.

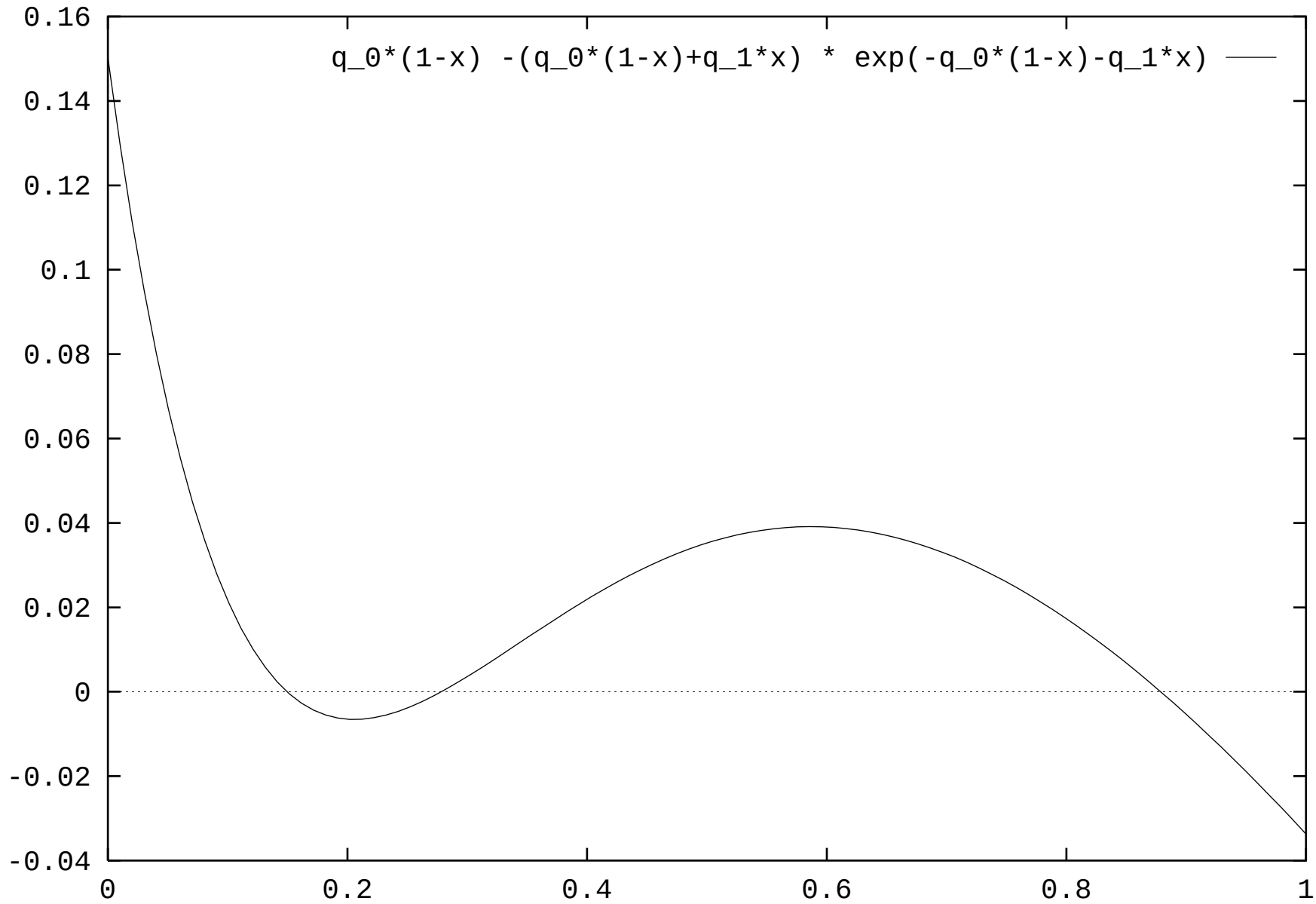
[26] S. R. S. Varadhan. *Large Deviations and Applications*. Society for Industrial and Applied Mathematics, Philadelphia, 1984.

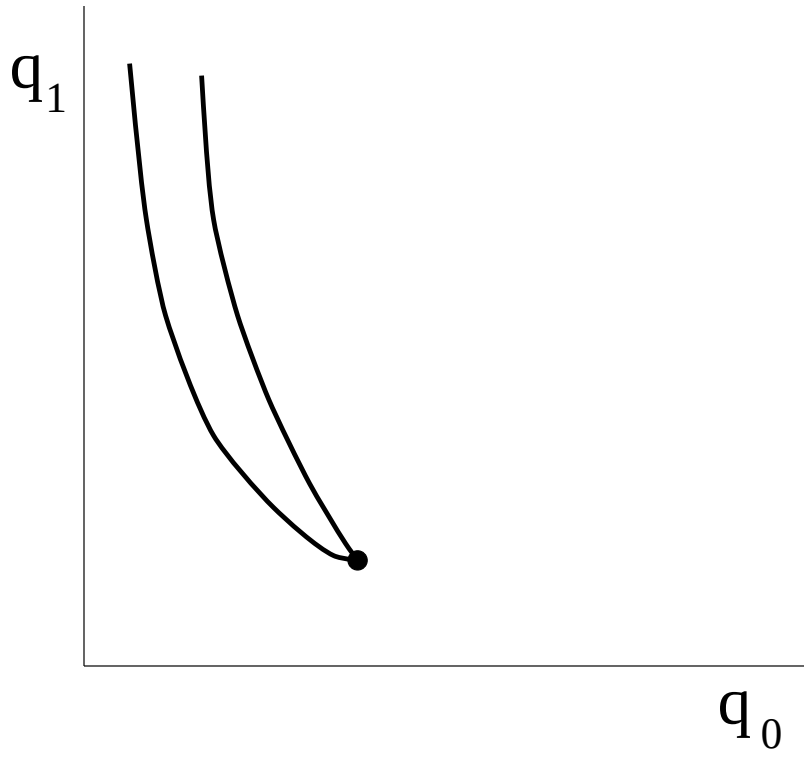
[27] A. D. Wentzell. Rough Limit Theorems for Large Deviations for Markov Processes. *Theory Probab. Appl.* **21** (1976), 227–242, 499–512.

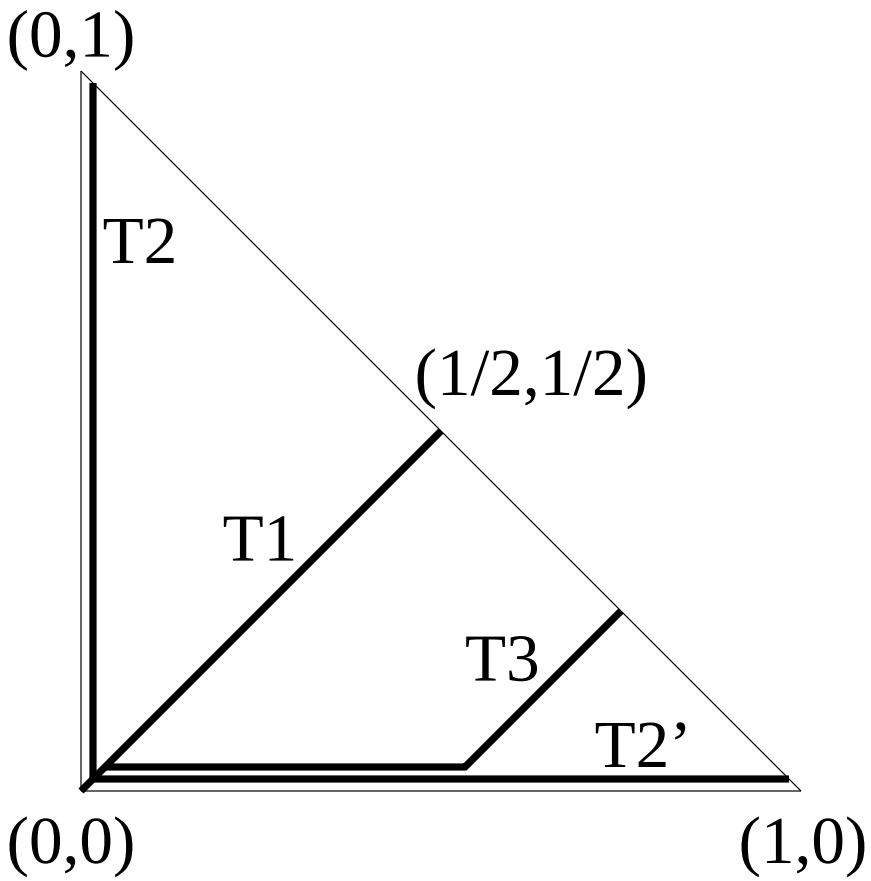
Figure 1: The expected drift velocity $\langle \xi(x) \rangle$ of the stochastic ALOHAnet model, as a function of normalized network state x. Model parameters are $q_0 = 0.43$ and $q_1 = 5.0$, as in Ref. [5].

Figure 2: An impressionistic sketch of the parameter space of the stochastic ALOHAnet model. Within the horn-shaped region the network is bistable; outside it, monostable. The tip of the horn is analogous to a statistical-mechanical critical point.

Figure 3: The triangular normalized state space of the colliding stacks model. Trajectory T1 is the most probable exit path when the function $p(x)$ is strictly decreasing, but if $p(x)$ is strictly increasing then T2 and T2$'$ are both MPEPs. Trajectory T3 is one of the uncountably many MPEPs which arise when $p(x)$ is independent of x.


q_0*(1-x) -(q_0*(1-x)+q_1*x) * exp(-q_0*(1-x)-q_1*x)
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
0
0.2
0.4
0.6
0.8
1


q_1
q_0


(0,1)
T2
(1/2,1/2)
T1
T3
T2’
(0,0)
(1,0)